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AN INTERSECTION NUMBER FOR THE PUNCTUAL HILBERT SCHEME OF A SURFACE

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ABSTRACT. We compute the intersection number between two cycles A and B of complementary dimensions in the Hilbert scheme H parameterizing subschemes of given finite length n of a smooth projective surface S. The $(n+1)$-cycle A corresponds to the set of finite closed subschemes the support of which has cardinality 1. The $(n-1)$-cycle B consists of the closed subschemes the support of which is one given point of the surface. Since B is contained in A, indirect methods are needed. The intersection number is $A.B = (-1)^{n-1}n$, answering a question by H. Nakajima.

1. Introduction

Let S be a smooth projective surface over an algebraically closed field k. For any natural number n let H_n denote the Hilbert scheme parameterizing finite subschemes of S of length n. It is smooth and projective of dimension $2n$.

Let $P \in S$ be a point, and let $M_n(P) \subseteq H_n$ be the closed reduced subvariety consisting of points which correspond to subschemes with support at P. Briançon proved that $M_n(P)$ is an irreducible variety of dimension $n-1$; see [2].

Denote by $M_n = \bigcup_{P\in S} M_n(P) \subseteq H_n$ the subvariety whose points correspond to subschemes with support in just one point. We may map M_n to S by sending a point of M_n to the point where the corresponding subscheme is supported. The fiber of this map over a point P being the variety $M_n(P)$, we see that M_n is irreducible of dimension $n+1$.

The subvarieties M_n and $M_n(P)$ are of complementary codimensions, and hence the product of their rational equivalence classes (or dual cohomology classes, if k is the field of complex numbers) defines an intersection number $\int_{H_n}[M_n]\cdot[M_n(P)]$. The main content of this note is the computation of that number. The result is:

1.1. Theorem. $\int_{H_n}[M_n]\cdot[M_n(P)] = (-1)^{n-1}n$.

One reason, pointed out to us by H. Nakajima, to be interested in these intersection numbers is the following. In case $k = \mathbf{C}$ is the field of complex numbers, Göttsche [5] computed the generating series $\sum_{m,n=0}^{\infty} \dim H^m(H_n,\mathbf{Q})t^n u^m$ and showed that it may be expressed in terms of classical modular forms. These forms are closely related to the trace of some standard representations of, respectively, the infinite Heisenberg algebra and the infinite Clifford algebra. In [6] Nakajima defined a representation of a product of these algebras, indexed over $H^*(S,\mathbf{Q})$,

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on the space $\bigoplus_{m,n=0}^{\infty} H^m(H_n, \mathbf{Q})$. He completely described this representation up to the determination of a series of universal constants c_n for $n = 1, 2, \dots,$ universal in the sense that they do not depend on the surface. He also proved that $c_n = \int_{H_n} [M_n] \cdot [M_n(P)]$. Hence we have

1.2. Theorem. *The Nakajima constants are given by* $c_n = (-1)^{n-1} n$.

2. Proof of the main theorem

The proof will be an inductive argument comparing the number c_n with c_{n+1}. To make this comparison, we shall make use of the "incidence variety", i.e., the closed, reduced subscheme of $H_n \times H_{n+1}$ given by $H_{n,n+1} = \{(\xi, \eta) \mid \xi \subseteq \eta\}$. It is known that $H_{n,n+1}$ is smooth and irreducible of dimension $2n+2$ (there are several proofs of this, see for example [3] or [7]).

There are obvious maps $f\colon H_{n,n+1} \to H_n$ and $g\colon H_{n,n+1} \to H_{n+1}$ induced by the projections. There is also a natural map $q\colon H_{n,n+1} \to S$ sending a pair (ξ, η) to the unique point where ξ and η differ. Let $Z_n \subseteq H_n \times S$ be the universal subscheme. It is finite and flat over H_n of rank n. Let $\pi_n\colon Z_n \to H_n$ denote the restriction of the projection. Furthermore, let $H'_n \subseteq H_n$ denote the open dense subset parameterizing local complete intersection subschemes, and put $Z'_n = \pi_n^{-1} H'_n$.

In the next section we will prove the following results, which shed light on the maps g and f.

2.1. Proposition. *The map* $g\colon H_{n,n+1} \to H_{n+1}$ *factors naturally as* $g = \pi_{n+1} \circ \psi$, *where* $\psi\colon H_{n,n+1} \to Z_{n+1}$ *is canonically isomorphic to* $\mathbf{P}(\omega_{Z_{n+1}})$. *In particular,* ψ *is birational and an isomorphism over* Z'_{n+1}, *and* g *is generically finite of degree* $n+1$.

2.2. Proposition. *The map* $\phi = (f, q)\colon H_{n,n+1} \to H_n \times S$ *is canonically isomorphic to the blowing up of* $H_n \times S$ *along* Z_n. *In particular, over* Z'_n, *the map* ϕ *is a* $\mathbf{P}^1$*-bundle.*

It follows that the fibers of f over local complete intersection subschemes $\xi \in H'_n$ are given as $f^{-1}(\xi) = \widetilde{S}(\xi)$, the surface S blown up along ξ. (This is also easy to see directly.)

The locus of pairs $(\xi, \eta) \in H_{n,n+1}$ where ξ and η have the same support is a divisor in $H_{n,n+1}$ which we denote by E. This is nothing but the exceptional divisor of the blowup morphism ϕ. On the fiber of f over a local complete intersection ξ, it restricts to the exceptional divisor of $\widetilde{S}(\xi)$.

Let $M_{n,n+1} = (g^{-1} M_{n+1})_{\mathrm{red}}$. We need the following strengthening of Briançon's result, also to be proved in the next section.

2.3. Proposition. $M_{n,n+1}$ *is irreducible, and* g *maps it birationally to* M_{n+1}. *In particular, all the* M_n *are irreducible, and the complete intersection subschemes form a dense open subset of* M_n.

Using the three propositions above, we have sufficient information to carry out the intersection computation. Let us summarize the situation in the following commutative diagram, where $B_n = \rho(M_{n,n+1})$:

$$\begin{tikzcd}
M_{n,n+1} \arrow[r, "\subseteq"] \arrow[d] & E \arrow[r, "j"] \arrow[d, "\rho"'] & H_{n,n+1} \arrow[r, "\gamma=(g,q)"] \arrow[d, "\phi=(f,q)"] & H_{n+1} \times S \arrow[d, "pr_2"] \\
B_n \arrow[r, "\subseteq"] \arrow[d] & Z_n \arrow[r, "i"] \arrow[d, "\pi_n"'] & H_n \times S \arrow[r, "pr_2"] \arrow[d, "pr_1"] & S \\
M_n \arrow[r, "\subseteq"] & H_n \arrow[r, equal] & H_n &
\end{tikzcd}$$

2.4. Lemma. *$g^*[M_{n+1}] = (n+1)\,[M_{n,n+1}]$ in $A^n(H_{n,n+1})$.*

Proof. Since $g^{-1}M_{n+1}$ is a multiple structure on $M_{n,n+1}$ by definition, and the codimensions of M_{n+1} and $M_{n,n+1}$ are the same, $g^*[M_{n+1}] = \ell\,[M_{n,n+1}]$ for some integer ℓ. Now use that $g_*[M_{n,n+1}] = [M_{n+1}]$ (by proposition 2.3) and the projection formula to get

$$(n+1)[M_{n+1}] = g_*g^*[M_{n+1}] = g_*(\ell\,[M_{n,n+1}]) = \ell\,[M_{n+1}],$$

proving that $\ell = n+1$. □

2.5. Lemma. *$[E] \cdot f^*[M_n] = n\,[M_{n,n+1}]$ in $A^n(H_{n,n+1})$.*

Proof. Consider first $[M_{n,n+1}]_E \in A_{n+2}(E)$. Let $h = \pi_n \circ \rho \colon E \to H_n$. Since $M_{n,n+1}$ is the support of $h^{-1}M_n$ and its codimension in E equals $\operatorname{codim}(M_n, H_n)$, we have that $h^*[M_n] = \ell\,[M_{n,n+1}]_E$, where ℓ is the multiplicity of $h^{-1}M_n$ at the generic point η of $M_{n,n+1}$. By proposition 2.2, ρ is smooth at η, so ℓ equals also the multiplicity of $\pi_n^{-1}M_n$ at the generic point $\rho(\eta)$ of B_n. But observing that B_n maps isomorphically to M_n, a similar argument as in the proof of lemma 2.4 shows that $\pi_n^*[M_n] = n\,[B_n]$, hence $\ell = n$.

We have shown that $h^*[M_n] = n\,[M_{n,n+1}]_E$ in $A_{n+2}(E)$. Apply j_* and the projection formula to get

$$n\,[M_{n,n+1}] = j_*h^*[M_n] = j_*j^*f^*[M_n] = [E] \cdot f^*[M_n].$$

□

Combining the two lemmas above, we get

$$\frac{1}{n+1}\,g^*[M_{n+1}] = \frac{1}{n}\,[E] \cdot f^*[M_n] \in A^n(H_{n,n+1}), \tag{2.1}$$

and exactly parallel reasoning shows that also

$$\frac{1}{n+1}\,g^*[M_{n+1}(P)] = \frac{1}{n}\,[E] \cdot f^*[M_n(P)] \in A^{n+2}(H_{n,n+1}). \tag{2.2}$$

We are now ready to prove theorem 1.1. Let F be a general fiber of f, for example corresponding to a reduced subscheme ξ. Clearly, $f^*[M_n] \cdot f^*[M_n(P)] = c_n\,[F]$. It is easy to see that $\int_F [E]^2 = -n$, and we get the following computation:

$$\begin{aligned}\frac{c_{n+1}}{n+1} &= \frac{1}{n+1}\int_{H_{n+1}} [M_{n+1}][M_{n+1}(P)] \\ &= \int_{H_{n,n+1}} \frac{1}{n+1}\, g^*[M_{n+1}]\cdot \frac{1}{n+1}\, g^*[M_{n+1}(P)] \quad \text{(proj. formula)} \\ &= \int_{H_{n,n+1}} \frac{1}{n}\,[E]f^*[M_n]\cdot\frac{1}{n}\,[E]f^*[M_n(P)] \qquad \text{((2.1) and (2.2))} \\ &= c_n\int_F \frac{1}{n^2}\,[E]^2 = \frac{-c_n}{n}.\end{aligned}$$

Now since trivially $c_1 = 1$, theorem 1.1 follows by induction.

3. The geometry of the incidence variety

The aim of this section is to prove propositions 2.1, 2.2, and 2.3 above. Some of the content of this section may be found in [4], but for the benefit of the reader we reproduce it here.

Consider a nested pair of subschemes $(\xi,\eta) \in H_{n,n+1}$, and let $P = q(\xi,\eta) \in S$ be the point where they differ. There are natural short exact sequence on S:

$$0 \to \mathcal{I}_\eta \to \mathcal{I}_\xi \to k(P) \to 0, \tag{3.1}$$

$$0 \to k(P) \to \mathcal{O}_\eta \to \mathcal{O}_\xi \to 0. \tag{3.2}$$

The first of these shows that the fiber $\phi^{-1}(\xi,P)$ is naturally identified with the projective space $\mathbf{P}(\mathcal{I}_\xi(P))$.

Dualizing (3.2), we arrive at another exact sequence:

$$0 \to \omega_\xi \to \omega_\eta \to k(P) \to 0, \tag{3.3}$$

and this shows that the fiber $\gamma^{-1}(\eta,P)$ maps naturally to $\mathbf{P}(\omega_\eta(P))$. Dualizing again, we see that (3.2) and hence ξ can be reconstructed from the right half of (3.3), so the map is an isomorphism.

It follows from (3.1) that

$$|\dim_k \mathcal{I}_\xi(P) - \dim_k \mathcal{I}_\eta(P)| \le 1. \tag{3.4}$$

(If $\mathcal{F}$ is a coherent sheaf, $\mathcal{F}(P)$ means $\mathcal{F}\otimes k(P)$.)

Note also that for any pair $(\xi,P) \in H_n \times S$, we have

$$\dim_k \mathcal{I}_\xi(P) = 1 + \dim_k \omega_\xi(P), \tag{3.5}$$

most easily seen using a minimal free resolution of the local ring $\mathcal{O}_{\xi,P}$ over $\mathcal{O}_{S,P}$.

The sequences (3.1) and (3.3) can be naturally globalized to the relative case of families of subschemes and points. This way one easily proves proposition 2.1, as well as the following lemma:

3.1. Lemma. *Let $\mathcal{I}_n$ denote the sheaf of ideals of Z_n in $\mathcal{O}_{H_n\times S}$. Then there is an isomorphism $H_{n,n+1} \simeq \mathbf{P}(\mathcal{I}_n)$ such that ϕ corresponds to the tautological mapping $\mathbf{P}(\mathcal{I}_n) \to H_n \times S$.*

We shall prove that the map $\phi\colon H_{n,n+1} \to H_n \times S$ is the blow up of $H_n \times S$ along the universal subscheme Z_n, by proving a general proposition on blowing up codimension two subschemes, and later verify its hypotheses in the case at hand.

Let W be any irreducible algebraic scheme and $Z \subseteq W$ a subscheme of codimension 2 whose ideal we denote by $\mathcal{I}_W$. We assume that $\mathcal{O}_Z$ is of local projective dimension 2 over $\mathcal{O}_W$. For any integer i let $W_i = \{w \in W \mid \dim_k \mathcal{I}_W(w) = i\}$. Let $\widetilde{W}$ be the blow up of W along Z.

There is an obvious map from $\widetilde{W}$ to $\mathbf{P}(\mathcal{I})$ due to the fact that $\mathcal{I}\mathcal{O}_{\widetilde{W}}$ is invertible. We shall see that under certain conditions this map is an isomorphism. (See also [1, Prop. 9].)

3.2. Proposition. *With the above hypothesis:*

(a) *Suppose that* $\operatorname{codim} W_i \geq i$ *for all* $i \geq 2$. *Then* $\mathbf{P}(\mathcal{I})$ *is irreducible and isomorphic to* $\widetilde{W}$.
(b) *If furthermore* Z *is irreducible and* $\operatorname{codim} W_i \geq i+1$ *for* $i \geq 3$*, then the exceptional divisor* E *is irreducible.*

Proof. The assumption on the local projective dimension gives an exact sequence

$$0 \to \mathcal{A} \xrightarrow{M} \mathcal{B} \to \mathcal{I} \to 0, \tag{3.6}$$

where $\mathcal{A}$ and $\mathcal{B}$ are vector bundles on W whose ranks are p and $p+1$ respectively, for some integer p. Locally the map M is given by a $(p+1)\times p$-matrix of functions on W, and the ideal $\mathcal{I}$ is locally generated by its maximal minors.

The sequence (3.6) induces a natural inclusion $\mathbf{P}(\mathcal{I}) \subseteq \mathbf{P}(\mathcal{B})$. In fact, letting $\varepsilon\colon \mathbf{P}(\mathcal{B}) \to W$ be the structure map and $\tau\colon \varepsilon^*\mathcal{B} \to \mathcal{O}_{\mathbf{P}(\mathcal{B})}(1)$ the tautological quotient, $\mathbf{P}(\mathcal{I})$ is defined in $\mathbf{P}(\mathcal{B})$ as the vanishing locus of the map $\tau \circ M$.

Consequently, any irreducible component of $\mathbf{P}(\mathcal{I})$ has dimension at least equal to $\dim(W)$.

Let $\phi\colon \mathbf{P}(\mathcal{I}) \to W$ be the structure map. Over W_i, the map ϕ is a $\mathbf{P}^{i-1}$-bundle. Hence by the assumption in (a), we have

$$\dim \phi^{-1} W_i \leq (\dim W - i) + (i-1) < \dim W$$

for all $i \geq 2$. It follows that $\phi^{-1}(W - Z)$ is dense in $\mathbf{P}(\mathcal{I})$, which is therefore irreducible.

To see that $\mathbf{P}(\mathcal{I})$ is isomorphic to the blow up, we remark that it follows from the resolution (3.6) that $\mathcal{I}\mathcal{O}_{\mathbf{P}(\mathcal{I})}$ is an invertible ideal. This gives a map from $\mathbf{P}(\mathcal{I})$ to $\widetilde{W}$, which over $\phi^{-1}(W-Z)$ is an inverse to the map from $\widetilde{W}$ to $\mathbf{P}(\mathcal{I})$ given above. As both spaces are irreducible, the two maps are inverses to each other.

Under the assumption in (b), it follows similarly that $\phi^{-1}W_2$ is dense in the exceptional locus $\phi^{-1}Z$, as all $\phi^{-1}W_i$ are of strictly lower dimension if $i \geq 3$. □

To finish the proof of proposition 2.2, we shall verify the conditions in proposition 3.2 for $Z = Z_n$ and $W = H_n \times S$.

Let $W_{i,n}$ be the set of points $(\xi, P) \in H_n \times S$ such that the ideal $\mathcal{I}_{Z_n}$ needs exactly i generators at (ξ, P). Equivalently,

$$W_{i,n} = \{(\xi, P) \in H_n \times S \mid \dim_k \mathcal{I}_\xi(P) = i\}. \tag{3.7}$$

We shall show by induction on n that $\operatorname{codim}(W_{i,n}, H_n \times S) \geq 2i - 2$, or equivalently, that

$$\dim W_{i,n} \leq 2n + 4 - 2i$$

for all $i, n \geq 1$. For $n = 1$ this is evidently satisfied for all i.

Assume that the inequality holds for a given n, and all $i \geq 1$. Then it follows that

$$\dim \phi^{-1} W_{j,n} \leq (2n + 4 - 2j) + (j - 1) = 2n + 3 - j \leq 2n + 4 - i$$

for all $j \geq i - 1$. By (3.4),

$$\gamma^{-1} W_{i,n+1} \subseteq \phi^{-1} W_{i-1,n} \cup \phi^{-1} W_{i,n} \cup \phi^{-1} W_{i+1,n}.$$

The fibers of γ over $W_{i,n+1}$ are $(i-2)$-dimensional. Hence

$$\dim W_{i,n+1} + (i - 2) = \dim \gamma^{-1} W_{i,n+1} \leq 2n + 4 - i$$

and hence $\dim W_{i,n+1} \leq 2(n + 1) + 4 - 2i$, as was to be shown.

Since $2i - 2 \geq i$ for $i \geq 2$ and $2i - 2 \geq i + 1$ for $i \geq 3$, the proof of proposition 2.2 is now complete. Note that we have also proved that the exceptional divisor E is irreducible.

Proof of proposition 2.3. The only hard part is to show that $M_{n,n+1}$ is irreducible. We will apply proposition 3.2 in the case where $W = M_n(P) \times S$ and $Z = Z_n \cap W$. As this intersection is proper, the condition on local projective dimension still holds in this case. Put $W'_{i,n} = W_{i,n} \cap W$. Similar reasoning as in the last proof gives the inequality $\operatorname{codim}(W'_{i,n}, W) \geq i + 1$ for all $n \geq 1$ and $i \geq 3$. Now note that the exceptional divisor $\phi^{-1} Z$ is nothing but $M_{n,n+1}$, which is therefore irreducible by proposition 3.2.

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